MULTILEVEL METHODS FOR TRANSPORT EQUATIONS IN DIFFUSIVE REGIMES

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SUMMARY

We consider the numerical solution of the single-group, steady state, isotropic transport equation. An analysis by means of the moment equations shows that a discrete ordinates S_N discretization in direction (angle) with a least squares finite element discretization in space does not behave properly in the diffusion limit. A scaling of the S_N equations is introduced so that the least squares discretization has the correct diffusion limit. For the resulting discrete system a full multigrid algorithm was developed.

1. INTRODUCTION

The single-group, steady state, isotropic form of the Boltzmann transport equation for one dimensional slab geometry is given by (Lewis and Miller [6])

$$\left\{ \begin{array}{l} \mu \dfrac{\partial \psi(x,\mu)}{\partial x} + \sigma_t \psi(x,\mu) - \sigma_s \dfrac{1}{2} \displaystyle\int_{-1}^{1} \psi(x,\mu')\, d\mu' = q(x,\mu) \\ \psi(a,\mu) = g_1(\mu) \quad \text{for} \quad \mu > 0 \\ \psi(b,\mu) = g_2(\mu) \quad \text{for} \quad \mu < 0 \end{array} \right\} \tag{1.1}$$

where $x \in [a,b]$ and $\mu \in [-1,1]$. When $\sigma_t \to \infty$ and $\frac{\sigma_s}{\sigma_t} \to 1$, which is the so called *diffusion limit*, this equation becomes singular. The limit operator $(I - P)$, where P denotes the operator $P\psi(x,\mu) \equiv \frac{1}{2}\int_{-1}^{1} \psi(x,\mu')d\mu'$, has in its nullspace all functions that are independent of angle μ.

Moreover, in this limit transport theory transitions into diffusion theory in the following way. Let ε be a small parameter. Substituting σ_t by $\frac{1}{\varepsilon}$, σ_s by $(\frac{1}{\varepsilon} - \varepsilon\sigma_a)$, where σ_a is $O(1)$, and scaling the right hand side by ε, equation (1.1) becomes

$$\left[\mu\frac{\partial}{\partial x} + \frac{1}{\varepsilon}I - \left(\frac{1}{\varepsilon} - \varepsilon\,\sigma_a\right) P\right] \psi(x,\mu) = \varepsilon q(x). \tag{1.2}$$

In addition, it is assumed that the external source q is independent of μ. As a consequence of this parameterization the diffusion limit is now equivalent to the limit $\varepsilon \to 0$. By expanding the solution

of (1.2) as

$$\psi(x,\mu) = \psi^0(x,\mu) + \sum_{j=1}^{\infty} \varepsilon^j \, \psi^j(x,\mu)$$

it can be shown (Larsen [2] [3], Larsen, Morel and Miller [4]) that some mean-free-paths away from the boundary the zeroth order term, $\psi^0(x,\mu)$, is independent of μ and is a solution of the following diffusion equation

$$-\frac{1}{3}\frac{d^2\phi(x)}{dx^2} + \sigma_a \phi(x) = q(x). \tag{1.3}$$

Thus, in the diffusion limit the solution of the transport equation will converge to the solution of a diffusion equation.

For the numerical solution of (1.1) it is important to find a discretization that has the same property; i.e., for diffusive regimes (σ_t large, $\frac{\sigma_s}{\sigma_t} \approx 1$) the difference scheme for the transport equation must approximate a diffusion operator.

In the last two decades a large amount of work was dedicated to developing special discretizations for the transport equation that have the correct behavior in the diffusion limit. Among them are the Diamond Difference scheme [6], the Linear Discontinuous scheme [1], and the Modified Linear Discontinuous scheme [5]. In the one-dimensional case their implementation is straightforward, but their extension to higher dimensions is difficult.

In this paper we try to develop a general framework for finding discretizations for the transport equation that have the correct behavior in the diffusion limit. In Section 2 we describe the discrete ordinates S_N discretization in angle and a least squares finite element discretization in space and discuss why this simple approach does not behave properly in the diffusion limit. A scaling technique for the transport equation is introduced in Section 3 that yields a least squares discretization with the proper diffusion limit. In Section 4 we present numerical results based on a full multigrid solver for the resulting discrete system. In Section 5 we draw conclusions and suggest further applications of the scaling technique.

2. DISCRETIZATION

For the discretization in angle we use the standard discrete ordinates S_N method. In the case of one-dimensional Slab geometry, this is a Galerkin discretization with normalized Legendre polynomials as basis. That means we are looking for a flux solution that has an expansion in the first N normalized Legendre polynomials,

$$\psi(x,\mu) = \sum_{l=0}^{N-1} \phi_l(x) \, P_l(\mu). \tag{2.1}$$

Since the normalized Legendre polynomials form an orthonormal basis for $L_2([-1,1])$, the moment coefficients ϕ_l are given by the following integral, which can be

written exactly as a sum by using a Gauss quadrature rule. We have

$$\begin{aligned} \phi_l(x) &= \frac{1}{2}\int_{-1}^{1} \psi(x,\mu)\, P_l(\mu)\, d\mu \\ &= \sum_{j=1}^{N} \omega_j\, \psi(x,\mu_j)\, P_l(\mu_j). \end{aligned} \tag{2.2}$$

Here μ_j denotes the Gauss quadrature points and ω_j denotes the Gauss quadrature weights.

By introducing the vector notation

$$\underline{\Psi} \equiv (\psi(x,\mu_1),\ldots,\psi(x,\mu_N))^\top; \qquad \underline{\Phi} \equiv (\phi_0(x),\ldots,\phi_{N-1}(x))^\top$$

and defining matrices T and Ω as

$$[T]_{i,j} \equiv P_{i-1}(\mu_j); \qquad \Omega \equiv \mathrm{diag}(\omega_1,\ldots,\omega_N)$$

the relationship (2.1) and (2.2) between the *flux* $\underline{\Psi}$ and the *moments* $\underline{\Phi}$ can be written as

$$\underline{\Phi} = T\Omega\underline{\Psi} \tag{2.3}$$

$$\underline{\Psi} = T^\top\underline{\Phi}. \tag{2.4}$$

As a result of the Galerkin discretization of (1.1) with the ansatz (2.1) we obtain the S_N equations

$$L\underline{\Psi} \equiv \varepsilon \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_N \end{pmatrix} \frac{\partial\underline{\Psi}}{\partial x} + I\,\underline{\Psi} - (1-\varepsilon^2\sigma_a)\,R\,\underline{\Psi} = \varepsilon^2\underline{q}, \tag{2.5}$$

where

$$R \equiv (1,\ldots,1)^\top\,(\omega_1,\ldots,\omega_N).$$

When we insert (2.4) into (2.5) and multiply by $T\Omega$ from the left, we get the moment equations

$$M\underline{\Phi} \equiv \left[\begin{array}{c|cccc} \varepsilon^2\sigma_a & \varepsilon b_0\frac{\partial}{\partial x} & 0 & 0 & \cdots \\ \hline \varepsilon b_0\frac{\partial}{\partial x} & 1 & \varepsilon b_1\frac{\partial}{\partial x} & 0 & \\ 0 & \varepsilon b_1\frac{\partial}{\partial x} & 1 & \varepsilon b_2\frac{\partial}{\partial x} & \\ \vdots & & \ddots & \ddots & \ddots \end{array}\right] \underline{\Phi} = \varepsilon^2\hat{\underline{q}} \tag{2.6}$$

with

$$b_j \equiv \frac{j+1}{\sqrt{4(j+1)^2-1}}.$$

Normally, the computations are done in the flux representation (2.5) since in this representation the boundary conditions are equal to simple Dirichlet boundary conditions. However, as we will see later, the moment equations are very useful for theoretical insight. In the following the flux operator is denoted by L and the moment operator by M.

For the spatial discretization of the S_N equations we use a least squares finite-element method based on the functional

$$F(\underline{\Psi}) \equiv \int_a^b \left\langle \Omega \left(L\underline{\Psi} - \underline{q} \right), L\underline{\Psi} - \underline{q} \right\rangle_{I\!R^N} dx, \tag{2.7}$$

and piecewise linear continuous elements η_k as basis functions for each component of $\underline{\Psi}$. In the following S^h denotes the finite dimensional space $S^h \equiv$ span $\{\eta_k\}$ and $(S^h)^N$ denotes the space of N-tuples whose elements are in S^h.

The advantage of this approach is that a least squares discretization converts the S_N equations, which are a coupled system of first order equations, into a self-adjoint variational formulation. Based on this variational formulation Multi-Level Projection Methods [7] can be applied in order to guide the development of a multigrid solver for the resulting discrete system.

Unfortunately, this discretization does not behave correctly in the diffusion limit. In order to explain this fact, we use the moment equations, since $\phi_0 = \psi$ in the diffusion limit. Further, it is easy to see by using the relationship (2.3), (2.4) and the identity $T^\top T\Omega = I$ that

$$\min_{\underline{\Psi} \in (S^h)^N} \int_a^b \left\langle \Omega \left(L\underline{\Psi} - \underline{q} \right), L\underline{\Psi} - \underline{q} \right\rangle_{I\!R^N} dx \iff$$

$$\min_{\underline{\Phi} \in (S^h)^N} \int_a^b \left\langle M\underline{\Phi} - \hat{\underline{q}}, M\underline{\Phi} - \hat{\underline{q}} \right\rangle_{I\!R^N} dx,$$

which justifies why it is also possible to look at the least squares discretization of the moment equations.

In the S_2 case with $\sigma_a = 0$, for example, a least squares discretization of the moment equations results in the following discrete system

$$\left(\begin{array}{c|c} \frac{\varepsilon^2}{3}(\eta', \eta') & \frac{\varepsilon}{\sqrt{3}}(\eta', \eta) \\ \hline \frac{\varepsilon}{\sqrt{3}}(\eta, \eta') & \frac{\varepsilon^2}{3}(\eta', \eta') + (\eta, \eta) \end{array} \right) \left(\begin{array}{c} \phi_0^h \\ \phi_1^h \end{array} \right) = \left(\begin{array}{c} 0 \\ \frac{\varepsilon^3}{\sqrt{3}}(\eta', q_0) \end{array} \right),$$

where, for example, (η, η) is a mass matrix and (η', η) a stiffnes matrix with elements

$$[(\eta, \eta)]_{k,l} \equiv \int_a^b \eta_k(x)\eta_l(x)dx, \qquad [(\eta', \eta)]_{k,l} \equiv \int_a^b \frac{d\eta_k(x)}{dx}\eta_l(x)dx$$

and

$$(\eta', q_0) \equiv \left(\ldots, \int_a^b \frac{d\eta_k(x)}{dx} q_0(x)dx, \ldots \right)^\top.$$

Forming the Schur complement we get the following equation for ϕ_0^h

$$\left\{ \frac{\varepsilon^2}{3}(\eta', \eta') - \frac{\varepsilon^2}{3}(\eta', \eta) \left[(\eta, \eta) + \frac{\varepsilon^2}{3}(\eta', \eta') \right]^{-1} (\eta, \eta') \right\} \phi_0^h$$

$$= -\frac{\varepsilon^4}{3}(\eta', \eta) \left[(\eta, \eta) + \frac{\varepsilon^2}{3}(\eta', \eta') \right]^{-1} (\eta', q_0). \tag{2.8}$$

For sufficient small ε we have

$$\left[(\eta,\eta)+\frac{\varepsilon^2}{3}(\eta',\eta')\right]^{-1}=(\eta,\eta)^{-1}-\frac{\varepsilon^2}{3}(\eta,\eta)^{-1}(\eta',\eta')(\eta,\eta)^{-1}+O(\varepsilon^4). \tag{2.9}$$

Plugging (2.9) into (2.8) and dividing by ε^2 leads to

$$\begin{aligned}&\left\{\frac{1}{3}\underbrace{[(\eta',\eta')-(\eta',\eta)(\eta,\eta)^{-1}(\eta,\eta')]}_{(*1)}\right.\\&\left.+\frac{\varepsilon^2}{9}(\eta',\eta)(\eta,\eta)^{-1}(\eta',\eta')(\eta,\eta)^{-1}(\eta,\eta')+O(\varepsilon^4)\right\}\phi_0^h\\&=-\frac{\varepsilon^2}{3}(\eta',\eta)(\eta,\eta)^{-1}(\eta',q_0)+O(\varepsilon^4).\end{aligned} \tag{2.10}$$

In the limit $\varepsilon\to 0$, the solution approaches a valid discretization for the diffusion equation (1.3) only if the term $(*1)$ vanishes identically. For piecewise linear, continuous basis elements this term does not cancel out and becomes the leading term in the equation. Consequently, in the diffusion limit (2.10) is an approximation for $\phi_0''=0$, which results in a linear solution, connecting the boundary conditions. In general, the term $(*1)$ does not have the proper behavior unless the mass matrix, (η,η), is lumped, that is, replaced by a diagonal matrix.

3. SCALING

A closer look at the moment equations (2.6) shows that this system is unbalanced. There are $O(\varepsilon^2)$, $O(\varepsilon)$ entries as well as $O(1)$ entries. The idea is to scale this system before the discretization.

First, let us consider the case $\sigma_a\neq 0$. In our inner product the adjoint moment operator, when homogeneous boundary conditions are assumed, is given by

$$M^*=\left[\begin{array}{c|cccc}\varepsilon^2\sigma_a & -\varepsilon b_0\frac{\partial}{\partial x} & 0 & 0 & \cdots\\ \hline -\varepsilon b_0\frac{\partial}{\partial x} & 1 & -\varepsilon b_1\frac{\partial}{\partial x} & 0 & \\ 0 & -\varepsilon b_1\frac{\partial}{\partial x} & 1 & -\varepsilon b_2\frac{\partial}{\partial x} & \\ \vdots & & \ddots & \ddots & \ddots\end{array}\right].$$

Scaling the moment equations by

$$S\equiv\begin{pmatrix}\frac{1}{\varepsilon\sigma_a} & & & \\ & \varepsilon & & \\ & & \varepsilon & \\ & & & \ddots\end{pmatrix} \tag{3.1}$$

and forming the normal equations results in

$$M^*SM\underline{\Phi}=\varepsilon^2M^*S\underline{\hat{q}}\iff$$

$$\left(\begin{array}{c|cccc} \varepsilon^3\sigma_a - \frac{\varepsilon^3}{3}\frac{\partial^2}{\partial x^2} & 0 & -\varepsilon^3 b_0 b_1 \frac{\partial^2}{\partial x^2} & 0 & \dots \\ \hline 0 & \varepsilon - \frac{\varepsilon b_0^2}{\sigma_a}\frac{\partial^2}{\partial x^2} - \varepsilon^3 b_1^2 \frac{\partial^2}{\partial x^2} & 0 & -\varepsilon^3 b_1 b_2 \frac{\partial^2}{\partial x^2} & \ddots \\ -\varepsilon^3 b_0 b_1 \frac{\partial^2}{\partial x^2} & 0 & \varepsilon - \varepsilon^3(b_1^2+b_2^2)\frac{\partial^2}{\partial x^2} & 0 & \ddots \\ 0 & -\varepsilon^3 b_1 b_2 \frac{\partial^2}{\partial x^2} & 0 & \varepsilon - \varepsilon^3(b_2^2+b_3^2)\frac{\partial^2}{\partial x^2} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{array}\right) \underline{\Phi}$$

$$= \begin{pmatrix} \varepsilon^3 \hat{q}_0 \\ \frac{-\varepsilon^2}{\sqrt{3}\sigma_a}\frac{\partial \hat{q}_0}{\partial x} \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{3.2}$$

Note that (3.2) already has the correct limit equations for the moments on its diagonal and all first order derivatives are eliminated.

Applying a Galerkin discretization to (3.2) and forming the Schur complement leads after division by ε^3 to the following discrete equation for ϕ_0^h

$$\left\{\sigma_a(\eta,\eta) + \frac{1}{3}(\eta',\eta') + O(\varepsilon^2)\right\}\phi_0^h = (\eta,\hat{q}_0) + O(\varepsilon^2),$$

which is a valid discretization of the corresponding diffusion equation (1.3) in the limit $\varepsilon \to 0$. When we define $\underline{b}_{k,j} \equiv \underline{e}_j \eta_k(x)$, where $\underline{e}_j$ denotes the j-th canonical unit vector of IR^N, we can write the Galerkin discretization of (3.2) as follows

$$\forall \underline{b}_{k,j} \int_a^b \left\langle M^* S\left(M\underline{\Phi} - \underline{\hat{q}}\right), \underline{b}_{k,j}\right\rangle_{IR^N} dx = 0. \tag{3.3}$$

Assuming homogeneous boundary conditions and splitting $S = \sqrt{S}\sqrt{S}$, (3.3) is equivalent to

$$\forall \underline{b}_{k,j} \int_a^b \left\langle \sqrt{S}\left(M\underline{\Phi} - \underline{\hat{q}}\right), \sqrt{S}M\underline{b}_{k,j}\right\rangle_{IR^N} dx = 0$$

$$\Longleftrightarrow \min_{\underline{\Phi}\in(S^h)^N} \int_a^b \left\langle \sqrt{S}\left(M\underline{\Phi} - \underline{\hat{q}}\right), \sqrt{S}\left(M\underline{\Phi} - \underline{\hat{q}}\right)\right\rangle_{IR^N} dx,$$

which is a least squares discretization of the moment equations, scaled by $\sqrt{S}$. Consequently, a least squares discretization of the moment equations, scaled by $\sqrt{S}$, also has the correct behavior in the diffusion limit.

Notice that (3.2) has the proper behavior for any $\sigma_a \neq 0$. The second equation contains $\frac{1}{\sigma_a}$ on both sides and yields the proper solution for ϕ_1 as $\sigma_a \to 0$.

On the other hand, if $\sigma_a = 0$ the scaling (3.1) cannot be applied. In this case, scaling the moment equations by

$$S_0 \equiv \begin{pmatrix} 1 & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \ddots \end{pmatrix}, \tag{3.4}$$

with

$$\alpha \equiv \varepsilon^p, \quad \text{where} \quad p > 2 \tag{3.5}$$

and forming the normal equations results in

$$\left(\begin{array}{c|cccc} -\varepsilon\alpha b_0^2 \frac{\partial^2}{\partial x^2} & -\varepsilon\alpha b_0 \frac{\partial}{\partial x} & -\varepsilon^2\alpha b_0 b_1 \frac{\partial^2}{\partial x^2} & 0 & \dots \\ \hline \varepsilon b_0 \frac{\partial}{\partial x} & \alpha - \varepsilon^2(b_0^2 + \alpha b_1^2)\frac{\partial^2}{\partial x^2} & 0 & -\varepsilon^2\alpha b_1 b_2 \frac{\partial^2}{\partial x^2} & \ddots \\ -\varepsilon^2\alpha b_0 b_1 \frac{\partial^2}{\partial x^2} & 0 & \alpha - \varepsilon^2\alpha(b_1^2 + b_2^2)\frac{\partial^2}{\partial x^2} & 0 & \ddots \\ 0 & -\varepsilon^2\alpha b_1 b_2 \frac{\partial^2}{\partial x^2} & 0 & \alpha - \varepsilon^2\alpha(b_2^2 + b_3^2)\frac{\partial^2}{\partial x^2} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{array}\right) \underline{\Phi}$$

$$= \begin{pmatrix} 0 \\ -\varepsilon^3 b_0 \frac{\partial \hat{q}_0}{\partial x} \\ 0 \\ \vdots \end{pmatrix}. \tag{3.6}$$

A Galerkin discretization of (3.6) leads to the following discrete system

$$\left(\begin{array}{c|c} \varepsilon^2\alpha b_0^2(\eta',\eta') & A_{01} \\ \hline A_{10} & A_{11} \end{array}\right) \underline{\Phi}^h = \begin{pmatrix} 0 \\ \varepsilon^3 b_0(\eta', \hat{q}_0) \\ 0 \\ \vdots \end{pmatrix}, \tag{3.7}$$

where

$$A_{01} \equiv (-\varepsilon\alpha b_0(\eta,\eta'), \varepsilon^2\alpha b_0 b_1(\eta',\eta'), 0, \dots)$$

$$A_{10} \equiv (\varepsilon\alpha b_0(\eta,\eta'), \varepsilon^2\alpha b_0 b_1(\eta',\eta'), 0, \dots)^\top$$

and $A_{11} \equiv$

$$\begin{pmatrix} \alpha(\eta,\eta) + \varepsilon^2(b_0^2 + \alpha b_1^2)(\eta',\eta') & 0 & \varepsilon^2\alpha b_1 b_2(\eta',\eta') & 0 & \dots \\ 0 & \alpha(\eta,\eta) + \varepsilon^2\alpha(b_1^2 + b_2^2)(\eta',\eta') & 0 & \varepsilon^2\alpha b_2 b_3(\eta',\eta') & \ddots \\ \varepsilon^2\alpha b_1 b_2(\eta',\eta') & 0 & \alpha(\eta,\eta) + \varepsilon^2\alpha(b_2^2 + b_3^2) & 0 & \ddots \\ 0 & \varepsilon^2\alpha b_1 b_2(\eta',\eta') & 0 & \alpha(\eta,\eta) + \varepsilon^2\alpha(b_3^2 + b_3^2) & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

For sufficient small ε we have

$$A_{11}^{-1} = \frac{1}{\varepsilon^2}\begin{pmatrix} \frac{1}{b_0^2}(\eta',\eta')^{-1} & & \\ & O(\frac{\varepsilon^2}{\alpha}) & \\ & & \ddots \end{pmatrix} + Q_1 + Q_2 + O(\frac{\alpha}{\varepsilon^2}),$$

with $Q_1 = O(\frac{\alpha}{\varepsilon^4})$ and $Q_2 = O(\frac{\varepsilon^2}{\alpha})$. Using this expansion when forming the Schur complement in (3.7) we get

$$\left\{\varepsilon^2\alpha b_0^2(\eta',\eta') - A_{01}A_{11}^{-1}A_{10}\right\}\phi_0^h = -A_{01}A_{11}^{-1}\begin{pmatrix}\varepsilon^3 b_0(\eta',\hat{q}_0)\\ 0\\ \vdots\end{pmatrix}$$

which after some algebra becomes

$$\begin{aligned}&\left\{\varepsilon^2\alpha b_0^2(\eta',\eta') - \alpha^2(\eta,\eta')(\eta',\eta')^{-1}(\eta,\eta') + O(\varepsilon^4\alpha)\right\}\phi_0^h\\ &\quad = -\varepsilon^2\alpha(\eta,\eta')(\eta',\eta')^{-1}(\eta',\hat{q}_0) + O(\alpha^2)\end{aligned} \tag{3.8}$$

Because of (3.5) we have $\frac{\alpha^2}{\varepsilon^2\alpha} \to 0$, so that (3.8) is a valid discretization of the corresponding diffusion equation in the diffusion limit. Using the same argument as above it follows that the least squares discretization of the moment equations, scaled by $\sqrt{S_0}$, also has the correct behavior in the diffusion limit.

As mentioned before, the computations are done in the flux representation. Therefore, we need to transfer the scaling of the moment equations to a scaling for the S_N equations. By means of the relationship (2.3) and (2.4) it follows that a least squares discretization of the moment equations, scaled by $\sqrt{S}$ is equivalent to a least squares discretization of the S_N equations, scaled by

$$T^\top\sqrt{S}T\Omega = \frac{1}{\sqrt{\varepsilon\sigma_a}}R + \sqrt{\varepsilon}\,(I-R)\,.$$

A further multiplication by $\sqrt{\varepsilon\sigma_a}$ leads to the following scaling in the case $\sigma_a \neq 0$

$$R + \varepsilon\sqrt{\sigma_a}\,(I-R)\,. \tag{3.9}$$

Similarly, in the case $\sigma_a = 0$ the least squares discretization of the moment equations, scaled by $\sqrt{S_0}$ with $p = 4$, is equivalent to a least squares discretization of the S_N equations, scaled by

$$R + \varepsilon^2\,(I-R)\,. \tag{3.10}$$

In order to avoid an "if else" in the computations it is possible to combine the scalings (3.9) and (3.10) to

$$R + \left(\varepsilon\sqrt{\sigma_a} + \varepsilon^2\right)(I-R)\,. \tag{3.11}$$

4. NUMERICAL RESULTS

For the solution of the discrete system that results from a least squares discretization of the S_N equations scaled by (3.11), a full multigrid in space algorithm with

- standard coarsening in space by doubling the mesh width,
- μ-line red-black smoothing,

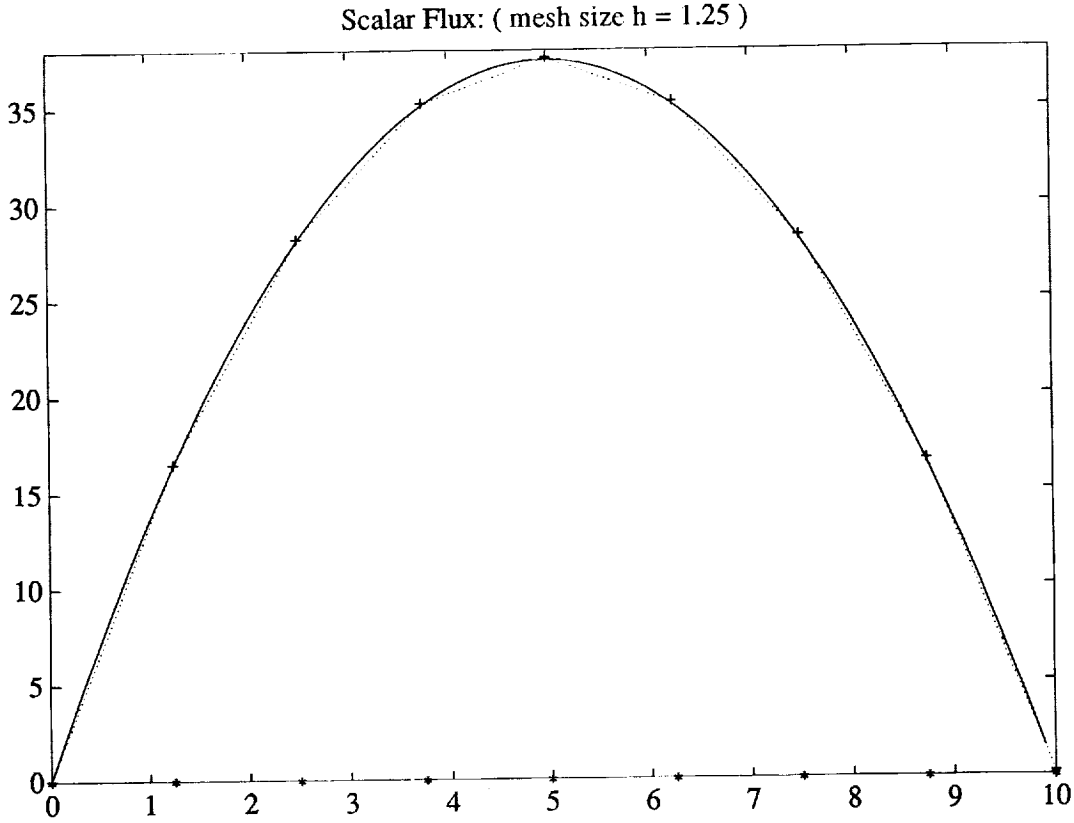

Figure 4.1: Scalar flux solution of problem (4.1)

- full weighting,
- linear interpolation,

was developed. The full multigrid process starts by solving the problem on the coarsest level and uses this solution as a starting guess for the next finer level, where a single V-cycle is performed. Recursively, the solution process proceeds from coarser levels to finer levels by halving each grid cell, using the coarse level solution as a starting guess and performing a single V-cycle on the next finer mesh. This algorithm yields V-cycle convergence rates that are below 0.09. Therefore, by performing one full multigrid V-cycle, a solution with an error on the order of the truncation error is obtained (cf. [7]).

As test problem we used the same problem that was used by Larsen, Morel and Miller in [4], which is shown below:

$$\left\{ \begin{array}{l} \mu_j \dfrac{\partial \psi_j}{\partial x} + 100\,\psi_j - 100 \displaystyle\sum_{\nu=1}^{N} \omega_\nu \psi_\nu = 0.01 \\ \psi_j(0) = 0 \quad \text{for} \quad \mu_j > 0 \\ \psi_j(10) = 0 \quad \text{for} \quad \mu_j < 0 \end{array} \right\}. \tag{4.1}$$

In our parametrization (1.2) this implies $\varepsilon = 0.01$, $\sigma_a = 0$ and $q = 1.0$. The exact solution of the corresponding diffusion equation is

$$\phi(x) = -\frac{3}{2}x^2 + 15x,$$

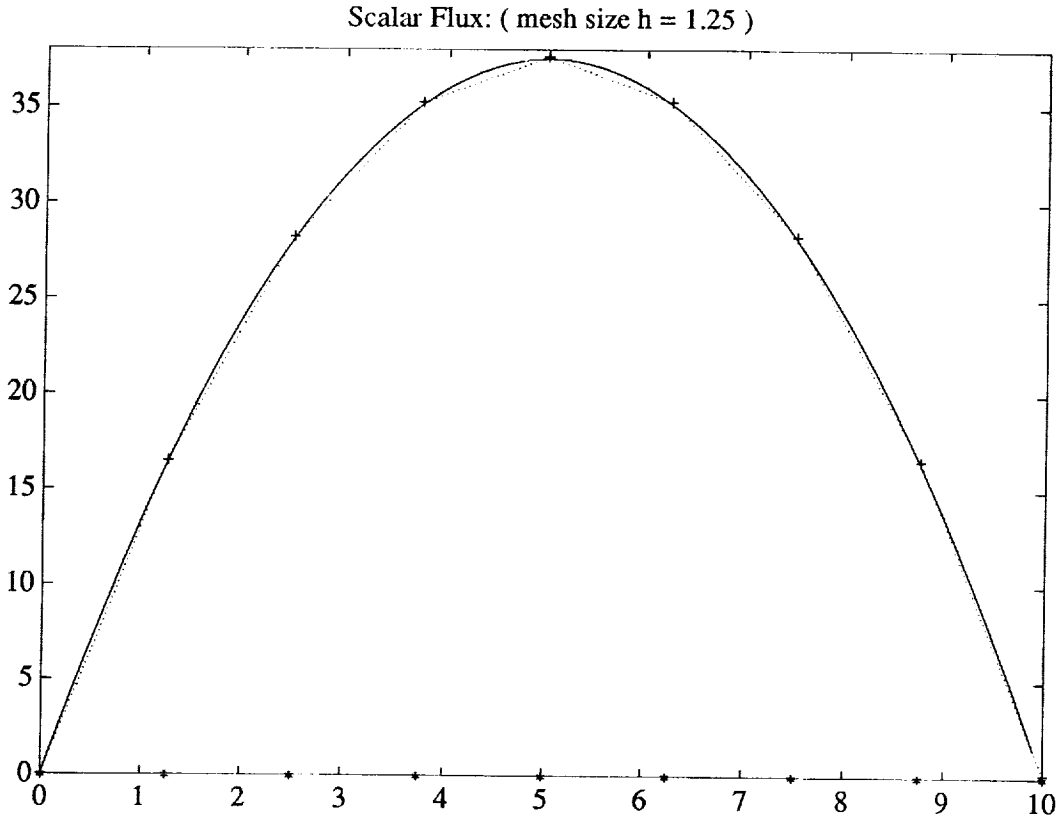

Figure 4.2: Scalar flux solution of problem (4.2)

which is plotted in Figure 4.1 in solid. The least squares solution of the scaled S_N equations, computed by one full multigrid V-cycle, is shown in Figure 4.1 by the crosses. We see that this is a very satisfactory result, especially when we take into consideration that we used a mesh size of 1.25, which is much larger than ε.

Finally, we mention that the least squares discretization of the S_N equations without scaling will give the zero solution for problem (4.1), indicated by the stars in Figure 4.1.

For the sake of completeness we present in Figure 4.2 the results for the test problem

$$\left\{ \begin{array}{l} \mu_j \dfrac{\partial \psi_j}{\partial x} + 100\,\psi_j - 99.99 \displaystyle\sum_{\nu=1}^{N} \omega_\nu \psi_\nu = 0.01 \left(1 - \frac{3}{2}x^2 + 15x\right) \\ \psi_j(0) = 0 \quad \text{for} \quad \mu_j > 0 \\ \psi_j(10) = 0 \quad \text{for} \quad \mu_j < 0 \end{array} \right\}, \tag{4.2}$$

where $\sigma_a = 1.0, \varepsilon = 0.01, q = 1 - \frac{3}{2}x^2 + 15x$. The exact solution of the corresponding diffusion equation is the same as for problem (4.1) and is again plotted in Figure 4.2 in solid. The least squares solution, computed by 1 full multigrid V-Cycle is given in Figure 4.2 by the crosses and the

solution for the least squares discretization of the S_N equations without scaling is given by the stars.

5. CONCLUSIONS

From the analysis in Section 3 and the numerical results presented in Section 4, we conclude that the least squares discretization of the scaled S_N equations has the proper behavior in the diffusion limit.

Further, we point out that the scaling can be used in the case of nonhomogeneous material, where σ_t or, equivalently, ε are discontinuous, because the equations are only scaled from the left side so that no derivatives are applied to the scaling operator.

Adaptive refinement can be combined with the full multigrid solver in a natural way. Areas of new refinement can be identified by examining the difference in the solution for two consecutive grids. This is especially important for nonhomogeneous material, where interior layers may exist.

Numerical results show that with a slightly different scaling both a Galerkin finite element formulation with piecewise linear elements and an Upwind Difference discretization of the S_N equations also have the correct diffusion limit. We believe that this scaling approach will result in a general framework for the development of discretizations that posses the correct diffusion limit.

Finally, we hope to apply the scaling techniques developed here to higher dimensional problems.

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